

Gravity duals to deformed SYM theories and generalized complex geometry

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ABSTRACT: We analyze the supersymmetry conditions for a class of $SU(2)$ structure backgrounds of Type IIB supergravity, corresponding to a specific ansatz for the supersymmetry parameters. These backgrounds are relevant for the AdS/CFT correspondence since they are suitable to describe mass deformations or beta-deformations of four-dimensional superconformal gauge theories. Using Generalized Complex Geometry we show that these geometries are characterized by a closed nowhere-vanishing vector field and a modified fundamental form which is also closed. The vector field encodes the information about the superpotential and the type of deformation - mass or beta respectively. We also show that the Pilch-Warner solution dual to a mass-deformation of $\mathcal{N} = 4$ Super Yang-Mills and the Lunin-Maldacena beta-deformation of the same background fall in our class of solutions.

KEYWORDS: Gauge-gravity correspondence, Flux compactifications.

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1. Introduction

Supergravity solutions with non-zero fluxes play an important role in any attempt to recover 4-dimensional physics from string theory, from string compactifications to the AdS/CFT correspondence. In the presence of type II Ramond-Ramond and/or Neveu-Schwarz fluxes, the internal six dimensional geometry back-reacts and is typically not Ricci-flat. In the last few years many attempts have been done to find a geometrical characterization of the internal manifolds analogue to the well-known Calabi-Yau condition in the absence of fluxes. The formalisms of G-structures and Generalised Complex Geometry have led to some progress in this direction. When applied to the AdS/CFT correspondence, the formalism of G-structures leads to a classification of the geometrical structure of known solutions but, more interestingly, can also be used to find new solutions.

The most elegant and best studied case is that of the structure group G being $SU(3)$. In this case it is possible to give a set of general conditions in order for the solutions to be $\mathcal{N} = 1$ supersymmetric and a full classification of these backgrounds is known: in type IIB the manifold has to be complex, while in type IIA it has to be twisted symplectic [1]. The geometry is fully characterized by a real two-form and a complex three-form. One of the two invariant forms (the two-form in IIA and the three-form in IIB) is conformally closed,

while the non-closure of the other cancels against the fluxes. The use of SU(3) structure allows to formulate the conditions for $\mathcal{N} = 1$ supersymmetry in a way [1, 2] which makes the task of finding explicit solutions easier. And this turned out to be particularly useful in the context of AdS/CFT correspondence. In [3] a family of regular SU(3)-structure equations was found describing the baryonic branch of the Klebanov-Strassler solution [4].

Less can be said about the case of SU(2) structure solutions (which in addition to the above mentioned forms admit a nowhere-vanishing vector field) where the considerable number of representations in the torsions and fluxes makes the analysis using G -structures less powerful. In particular, it has been shown that IIB backgrounds with SU(2) structure are no longer required to be complex in order to preserve supersymmetry [5]. In the context of the AdS/CFT correspondence various important solutions are characterized by SU(2) structures. In fact, while conformal backgrounds of the form $AdS_5 \times H$, with H a Sasaki-Einstein manifold, and the corresponding non conformal backgrounds obtained by adding fractional branes are described by SU(3) structures, massive and marginal deformations of these conformal theories are typically characterized by SU(2) structures and their geometry is still poorly understood. It is one of the purposes of this paper to start a detailed analysis of the conditions of supersymmetry related to SU(2) structures. Having in mind applications to the AdS/CFT correspondence, we will consider the case of type IIB solutions with non compact internal manifolds. However, the geometrical characterization of the SU(2) backgrounds described in this paper also have applications to the compact case.

We will make use of the language of Generalised Complex Geometry (GCG), which is a convenient conceptual framework for describing the $\mathcal{N} = 1$ geometries. The basic objects here are *pure spinors*, formal sums of even or odd forms, whose existence imposes certain topological conditions on the sum of the tangent and cotangent bundles of the internal manifold. In this language preservation of $\mathcal{N} = 1$ supersymmetry reduces to a pair of differential conditions on the pure spinors which are somewhat schematically:

$$d_H \Phi_- = 0, \tag{1.1}$$

$$d_H \Phi_+ = *F. \tag{1.2}$$

Here Φ_+ (Φ_-) is the even (odd) pure spinor and F is the formal sum of all RR fluxes. We will give their explicit expressions in section 2. It is important that the pure spinors satisfy algebraic compatibility conditions in order to define a Riemannian metric on the internal space. Manifolds admitting a closed pure spinor (which as we see is one of the necessary conditions for $\mathcal{N} = 1$ supersymmetry) are called Generalized Calabi-Yau (GCY) manifolds.

In this paper we discuss the supersymmetry equations (and Bianchi identities) for a particularly interesting class of SU(2)-structure backgrounds. We will show that with a particular ansatz for the supersymmetry spinors the conditions for $\mathcal{N} = 1$ supersymmetry give a very simple set of equations for the SU(2) invariant forms and the fluxes. As all SU(2) structure backgrounds, these solutions are characterized by the existence of a conformally closed vector. In addition to that, even if the metric is not Kähler, it is possible to define a modified (1,1)-form which is conformally closed. We will show that two well

known solutions in the context of the AdS/CFT correspondence, the Pilch-Warner solution (PW) [6], describing a massive deformation of $\mathcal{N} = 4$ SYM with an IR fixed point, and the Lunin-Maldacena (LM) [7], describing the marginal β -deformation of $\mathcal{N} = 4$ SYM, belong to this class of backgrounds. More generally, we expect to be able to characterize in terms of these special SU(2) structure all other warped AdS_5 solutions with fluxes corresponding to (IR limits of) massive deformations and β -deformations of conformal field theories¹. In this paper we will be mostly concerned with AdS_5 solutions, but our equations have applications to non conformal solutions as well. One particularly important case of non conformal backgrounds which should belong to the special class of solutions considered in this paper is given by the general massive deformations of $\mathcal{N} = 4$ SYM, $\mathcal{N} = 1^*$.

We will also study in detail the introduction of D3-brane probes in the GCY backgrounds. The analysis of the supersymmetry conditions for a probe determines a (possibly empty) sub-variety of the internal manifold that is in correspondence with the (mesonic) moduli space of vacua of the dual gauge theory. We will show that all supergravity solutions dual to a gauge theory with a non trivial moduli space of vacua necessarily belong to the special class of SU(2) structures considered here.

The paper is organized as follows. In section 2, we review the geometrical characterization of SU(2) structure backgrounds and we introduce the general tools that will be used in the rest of the paper. In section 3 we specialize to a particular spinorial ansatz and we write a very simple set of equations involving the SU(2) invariant forms and the fluxes. In section 4 we discuss the geometry of the PW flow and the massive deformations of conformal theories. We generalize a very simple class of type IIB supersymmetric solutions found in [8]: a class of complex manifolds and associated fluxes which solve the supersymmetry conditions of type IIB and are completely characterized by the existence of a generalized Kähler potential. These backgrounds can be used to describe massive flows to IR fixed points. In this context we give some general conditions specifying AdS_5 solutions. In section 5 we discuss the geometry of the LM solutions and of (marginal) β -deformations of conformal theories. We show that quite generally the action of T-duality on a Calabi-Yau background leads to an SU(2) structure of the special case considered in this paper. Finally, appendix A contains the set of supersymmetry conditions for the most general SU(2) structure ansatz and appendix B collects the formulae for the T-duality which are used for the LM solution.

2. SU(2) structure backgrounds

In this paper we are interested in solutions of IIB supergravity of warped type

$$ds^2 = e^{2A} ds_4^2 + ds_6^2, \tag{2.1}$$

where the internal manifold has SU(2) structure.

¹A general set of equations for AdS_5 solutions in type IIB supergravity have been written in [9].

An SU(2) structure manifold is characterized by the existence of two globally defined never-vanishing spinors which are never parallel

$$\eta_+ \quad \chi_+ = \frac{1}{2}z \cdot \eta_-, \quad (2.2)$$

where η_- is the complex conjugate of η_+ and $z \cdot$ denotes the Clifford multiplication by the one-form $z_m \gamma^m$.

Consequently the ten-dimensional supersymmetry parameters can be written as

$$\begin{aligned} \epsilon_1 &= \zeta_+ \otimes \eta_+^1 + \zeta_- \otimes \eta_-^1, \\ \epsilon_2 &= \zeta_+ \otimes \eta_+^2 + \zeta_- \otimes \eta_-^2, \end{aligned} \quad (2.3)$$

where ζ_{\pm} is a $4d$ chiral spinor ($\zeta_+^* = \zeta_-$) and the $6d$ chiral spinors $\eta_{\pm}^{(i)}$ are related to the SU(2) structure spinors by

$$\begin{aligned} \eta_{1+} &= a\eta_+ + b\chi_+, \\ \eta_{2+} &= x\eta_+ + y\chi_+, \end{aligned} \quad (2.4)$$

with a, b, x and y complex functions on the internal manifold. For the familiar SU(3) structure case corresponding to a Calabi-Yau manifold one has $x = -ia$ and $b = y = 0$.

An alternative definition of SU(2) structure which will be useful in the following is given in terms of globally defined never-vanishing bilinears in the spinors (2.2)

$$z = -2\chi_-^\dagger \gamma_m \eta_+ dx^m, \quad (2.5)$$

$$j = \frac{i}{2}\chi_+^\dagger \gamma_{mn} \chi_+ dx^m \wedge dx^n - \frac{i}{2}\eta_+^\dagger \gamma_{mn} \eta_+ dx^m \wedge dx^n, \quad (2.6)$$

$$\omega = -i\chi_+^\dagger \gamma_{mn} \eta_+ dx^m \wedge dx^n. \quad (2.7)$$

Here z is a complex 1-form, j and ω a real 2-form and a complex (2,0)-form satisfying

$$\omega \wedge j = 0, \quad (2.8)$$

$$j^2 = \frac{1}{2}\omega \wedge \bar{\omega}, \quad (2.9)$$

$$z \lrcorner j = z \lrcorner \omega = 0. \quad (2.10)$$

Each of the spinors η_+ and χ_+ defines an almost complex structure compatible with the metric. The associated (1,1)-forms are given by

$$\begin{aligned} J &= -\frac{i}{2}\eta_+^\dagger \gamma_{mn} \eta_+ dx^m \wedge dx^n = +j + \frac{i}{2}z \wedge \bar{z}, \\ \tilde{J} &= -\frac{i}{2}\chi_+^\dagger \gamma_{mn} \chi_+ dx^m \wedge dx^n = -j + \frac{i}{2}z \wedge \bar{z}, \end{aligned} \quad (2.11)$$

respectively. More generally, the SU(2) structure determines an entire U(1) family of almost complex structures compatible with the metric. The corresponding (1,1)-forms are constructed in terms of the normalized spinor

$$\xi = \cos \delta \eta + \sin \delta \chi \quad (2.12)$$

as $J_\xi = -(i/2)\xi_+^\dagger \gamma_{mn} \xi_+ dx^m \wedge dx^n$.

As shown in [2], by tensoring the supersymmetry parameters on the internal manifold, $\eta_\pm^{1,2}$, it is possible to define formal sums of even and odd forms respectively

$$\Phi_+ = \eta_+^1 \otimes \eta_+^{2\dagger}, \tag{2.13}$$

$$\Phi_- = \eta_+^1 \otimes \eta_-^{2\dagger} \tag{2.14}$$

which are interpreted as pure spinors of $Cliff(6, 6)$ in the context of Generalised Complex Geometry [10, 11].

For the choice of η^1 and η^2 in (2.4), the explicit form of the pure spinors reads

$$\Phi_+ = \frac{1}{8} \left[a\bar{x}e^{-ij} + b\bar{y}e^{ij} - i(a\bar{y}\omega + \bar{x}b\bar{\omega}) \right] e^{z\bar{z}/2}, \tag{2.15}$$

$$\Phi_- = \frac{1}{8} \left[i(by\bar{\omega} - ax\omega) + (bx e^{ij} - aye^{-ij}) \right] z. \tag{2.16}$$

Following [2], one can check explicitly that both Φ_+ and Φ_- are annihilated by six combinations of gamma-matrices and thus are pure. Since they have three annihilators in common, they are also compatible. We would like to point out that the ansatz for the spinors and consequently the form of the pure spinors are slightly different from those used in [2]. The two choices are related by a rotation of $SU(2)$ structure that sets b to zero in (2.4). The form (2.15) seems to be more suitable to describe the type of spinor ansatz that appears in the supergravity solutions dual to mass deformations that we want to analyze in this paper. Notice also that, in this form, the limits where the two spinors $\eta_{1,2}$ are always parallel ($SU(3)$ structure) and always orthogonal ($SU(2)$ structure) are both smooth.

As already mentioned in the Introduction, the supersymmetry variations of the supergravity fermions can then be re-expressed as two equations for the two pure spinors

$$\begin{aligned} e^{-2A+\phi}(d - H\wedge)(e^{2A-\phi}\Phi_-) &= 0, \\ e^{-2A+\phi}(d - H\wedge)(e^{2A-\phi}\Phi_+) &= dA \wedge \bar{\Phi}_+ + \frac{e^\phi}{16} \left(a_- F_{\text{IIB}} - ia_+ * F_{\text{IIB}} \right), \end{aligned} \tag{2.17}$$

with $F_{\text{IIB}} = F_1 + F_3 + F_5$ and $a_\pm = |a|^2 + |b|^2 \pm (|x|^2 + |y|^2)$. The functions a, b, x, y are related to the norms of the pure spinors and satisfy

$$\begin{aligned} d(|a|^2 + |b|^2) &= (|x|^2 + |y|^2) dA, \\ d(|x|^2 + |y|^2) &= (|a|^2 + |b|^2) dA. \end{aligned} \tag{2.18}$$

By expanding into forms of definite degree, a set of necessary conditions for $\mathcal{N} = 1$ solutions can be derived. The complete set of equations corresponding to a generic ansatz (2.4) is reported in appendix A.

The conditions for the existence of supersymmetric branes in a generalized Calabi-Yau geometry have been studied in [12]. A general requirement is that the norms of the two spinors $\eta_{1,2}$ are equal. In our notations, this translates into $a_- = 0$, which, combined with (2.18), gives

$$(|a|^2 + |b|^2) = (|x|^2 + |y|^2) = e^A. \tag{2.19}$$

This condition has to be satisfied by all backgrounds arising as the near horizon geometry of systems of branes.

We will be interested in adding D-branes to the background, and, in particular, space-time filling D3 branes. In the context of the AdS/CFT solutions that we will consider, a supersymmetric D3 brane probes the (mesonic) moduli space of the dual gauge theory ². We are in fact interested in backgrounds which originate from a stack of N D3 branes. The mesonic moduli space of vacua is in correspondence with the supersymmetric distributions of N branes in different points of the internal manifold. In the familiar case of D3 branes probing a singular Calabi-Yau cone (with dual background $AdS_5 \times H$, where H is the Sasaki-Einstein base of the cone) the moduli space is just the symmetrized product of N copies of the Calabi-Yau manifold. In more general deformed backgrounds, fluxes can alter the supersymmetry conditions. In particular they can introduce a superpotential for the probe that may reduce the moduli space.

The supersymmetric conditions for space-time filling Dp-brane can be expressed in terms of the pure spinors (2.15) as [12]

$$\begin{aligned} \text{Im}(i\Phi_+) \wedge e^{F-B}|_{\text{top}} &= 0, \\ ((dx^n + g^{nm}l_m)\Phi_-) \wedge e^{F-B}|_{\text{top}} &= 0, \end{aligned} \tag{2.20}$$

where F is the world-volume gauge field. These equations, in the case of a D3 brane, become

$$\begin{aligned} \text{Im}(i\Phi_+)|_{(0)} &= 0, \\ \Phi_-|_{(1)} &= 0, \end{aligned} \tag{2.21}$$

where $\Phi_{\pm}|_{(k)}$ denotes the k -form component of the spinor. The two constraints can be interpreted as a D-term and an F-term conditions for the probe brane. For example, it has been shown [12] that the superpotential for the probe brane is given by

$$dW = -ie^{2A-\varphi}\Phi_-|_{(1)} = \frac{i}{8}e^{2A-\varphi}(ay - bx)z. \tag{2.22}$$

More explicitly, from equation (2.15) we obtain the D-term and an F-term conditions for a supersymmetric D3 brane probe

$$\text{Re}(a\bar{x} + b\bar{y}) = 0, \tag{2.23}$$

$$(bx - ay) = 0. \tag{2.24}$$

The conditions for supersymmetry (2.17) imply ³

$$d(e^{A-\varphi}(a\bar{x} + b\bar{y})) = 0. \tag{2.25}$$

²It should be noted that, in general, beside the mesonic branch there may exist other vacua such as baryonic flat directions or special non-abelian vacua related to a particular form of the superpotential. These however are not interpreted as D3 branes moving in the internal manifold.

³Just take the real part of the one-form component of the equation for Φ_+ – equation (A.4) in appendix A.

The quantity $(a\bar{x} + b\bar{y})$ is therefore non-vanishing on the internal six-manifold. It follows that the D-term condition (2.23) is satisfied either everywhere or nowhere. In the cases where the condition (2.23) is not satisfied, the moduli space is empty. When (2.23) is satisfied, the F-term condition (2.24) will select the sub-manifold where a supersymmetric probe can freely move. This sub-manifold is in correspondence with the mesonic moduli space of vacua of the dual gauge theory.

3. A class of SU(2) structures

In this paper we will focus on a specific form for the spinorial ansatz (2.4) which gives rise to a particularly simple set of supersymmetry conditions.

We will consider the ansatz

$$\begin{aligned}\eta_{1+} &= a\eta_+ + b\chi_+, \\ \eta_{2+} &= -i(a\eta_+ - b\chi_+),\end{aligned}\tag{3.1}$$

and, using equation (2.19), we will parametrize

$$a = ix = ie^{A/2} \cos \phi e^{i\alpha},\tag{3.2}$$

$$b = -iy = -ie^{A/2} \sin \phi e^{i\beta}.\tag{3.3}$$

With this choice the supersymmetry conditions become very simple. Some of the equations only contain the geometric data of the solution

$$d\left(e^{3A-\varphi} e^{i(\alpha+\beta)} \sin 2\phi z\right) = 0,\tag{3.4}$$

$$d\left[e^{2A-\varphi} \left(j + \frac{i}{2} \cos 2\phi z \wedge \bar{z}\right)\right] = 0.\tag{3.5}$$

The other equations mix the geometry and the fluxes

$$e^{-4A+\varphi} d\left(e^{4A-\varphi} \cos 2\phi\right) = -e^\varphi * F_5,\tag{3.6}$$

$$e^{-4A+\varphi} d\left(e^{4A-\varphi} \sin 2\phi \operatorname{Im}\hat{\omega}\right) = \cos 2\phi H - e^\varphi * F_3,\tag{3.7}$$

$$\left[d\left(\frac{\cos^2 \phi \hat{\omega} + \sin^2 \phi \bar{\hat{\omega}}}{\sin 2\phi}\right) + iH\right] \wedge z = 0,\tag{3.8}$$

$$e^{-2A+\varphi} d\left(e^{2A-\varphi} \sin 2\phi \operatorname{Im}\hat{\omega} \wedge z \wedge \bar{z}\right) + 2iH \wedge \left(j + \frac{i}{2} \cos 2\phi z \wedge \bar{z}\right) = 0,\tag{3.9}$$

$$e^{-4A+\varphi} d\left[e^{4A-\varphi} (\cos 2\phi j^2 + ij \wedge z \wedge \bar{z})\right] + 2 \sin 2\phi H \wedge \operatorname{Im}\hat{\omega} = e^\varphi * F_1\tag{3.10}$$

where $\hat{\omega} = e^{i(\alpha-\beta)}\omega$.

The geometrical conditions (3.4), (3.5) give the following description of the six-dimensional internal space. It is convenient to define the Weyl rescaled six-dimensional metric

$$ds_6^2 = e^{-2A+\varphi} d\tilde{s}_6^2\tag{3.11}$$

which is the one seen by a D3 brane probing the background. The rescaled internal manifold is characterized by a conformally closed vector z . In the almost complex structure determined by η_+ , the metric is not Kähler, since $dJ \neq 0$. However the modified two-form

$$\hat{J} = J - i \sin^2 \phi z \bar{z} = j + \frac{i}{2} \cos 2\phi z \bar{z} \tag{3.12}$$

is closed.

We can characterize our class of solutions as corresponding to the $\mathcal{N} = 1$ backgrounds with a nontrivial moduli space for D3-brane probes. This can be seen as follows. Given an $SU(2)$ solution corresponding to the spinor ansatz (2.4), we can still make a redefinition of η_+ as in (2.12). The only effect would be a redefinition of the almost complex structure (alternatively a redefinition of J , or j, ω) we choose in order to write our equations. There exist choices where the equations simplify. However, not every spinor ansatz (2.4) can be reduced to the form considered in this section. A generic spinor ansatz (2.4) can be brought to the form (3.1) by a redefinition of η_+ if and only if

$$\text{Re}(a\bar{x} + b\bar{y}) = 0. \tag{3.13}$$

Equation (3.13) is exactly the D-term condition for the existence of a moduli space of vacua for probe D3 branes. We recall from the previous section that, for a generic supersymmetric background, the condition (3.13) is satisfied in all points of the internal manifold or nowhere. This means that the class of $SU(2)$ structures we have just discussed contains at least all $\mathcal{N} = 1$ backgrounds admitting a non trivial mesonic moduli space of vacua.

We can also consider the explicit form of the moduli space of vacua for this class of $SU(2)$ backgrounds. The F-term condition (2.24) simplifies to $ab = 0$ or

$$\sin 2\phi = 0. \tag{3.14}$$

This condition selects a (possibly empty) sub-manifold of the internal space corresponding to the moduli space of supersymmetric vacua of the dual gauge theory.

4. Massive deformations of conformal gauge theories

An interesting class of $SU(2)$ structure backgrounds is provided by the duals of massive deformations of conformal gauge theories.

The typical example is $\mathcal{N} = 4$ Super Yang-Mills to which we can add a general supersymmetric mass deformation

$$\int d\theta^2 dx^4 m_{ij} \Phi_i \Phi_j. \tag{4.1}$$

On the supergravity side the massive deformation corresponds to a non-zero value of the complex 3-form $G_3 = F_3 - ie^{-\varphi} H$. It is known that, by deforming $\mathcal{N} = 4$ SYM with a mass term for a single adjoint, the theory flows to a fixed point. This follows from a standard argument due to Leigh and Strassler [13]. The deformed theory has a superpotential

$$g\Phi_3[\Phi_1, \Phi_2] + m\Phi_3^2 \tag{4.2}$$

which, after integrating out the massive field, becomes $-\frac{g^2}{4m}[\Phi_1, \Phi_2]^2$. The conditions of conformal invariance, corresponding to the vanishing of the beta-function and the requirement for the superpotential to have dimension three, combined with the obvious SU(2) symmetry of the theory, require

$$\Delta_{\Phi_{1,2}}(g, m) = \frac{3}{4}. \tag{4.3}$$

This equation for the parameters g and m determines a line of IR fixed points⁴. The IR conformal field theory has a complex two dimensional moduli space of vacua parametrized by $\Phi_{1,2}$. The supergravity solution corresponding to this flow has been studied in [6] and it will be referred to as the PW flow. It was originally obtained using five-dimensional gauged supergravity and then lifted to ten dimensions. The PW solution is complex, with constant dilaton and non trivial profile for the antisymmetric three-form fields of type IIB. It reduces in the IR to a warped AdS background with three-form fluxes. At the IR fixed point, the constant dilaton parametrizes the line of conformal field theories.

It has been shown in [8] that the supersymmetry parameter $\epsilon = \epsilon_1 + i\epsilon_2$ satisfies a dielectric-like projection

$$\epsilon = \cos \phi \Gamma_{0123} \epsilon + i \sin \phi \Gamma_{0123xy} \epsilon^*, \tag{4.4}$$

where x, y are internal directions. Supersymmetry parameters in this class can be reduced to the ansatz (3.1). A solution to the projection (4.4) is indeed given by a modification of the supersymmetry parameter of the undeformed background, $\epsilon_0 = \xi_+ \otimes \eta_+^0$,

$$\epsilon = \cos \phi \epsilon_0 + i \Gamma_{xy} \sin \phi \epsilon_0^*. \tag{4.5}$$

Here η^0 determines the complex structure of the undeformed background. In the case of $\mathcal{N} = 4$ SYM we are dealing with \mathbb{C}^3 ; more generally, we may consider a Calabi-Yau cone corresponding to an $\mathcal{N} = 1$ superconformal gauge theory. In all cases, $\Gamma_{xy}(\eta_+^0)^*$ can be rewritten as $(\bar{z}/2) \cdot \eta_+^0$ for a suitable complex vector z . From equation (2.3) it follows that

$$\begin{aligned} \eta_+^1 &\sim \cos \phi \eta_+^0 + \sin \phi \frac{z}{2} \cdot \eta_-^0, \\ \eta_-^2 &\sim -i(\cos \phi \eta_+^0 - \sin \phi \frac{z}{2} \cdot \eta_-^0), \end{aligned} \tag{4.6}$$

corresponding to the ansatz (3.1). Thus the mass deformation in the field theory selects one complex direction in the internal space corresponding to the 1-form z .

The full flow from $\mathcal{N} = 4$ to the LS fixed point is then described by an SU(2) structure with a spinor ansatz of the form (3.1). In the following we will analyze in detail the structure of the PW flow and, more generally, of deformations with three-form G induced by a massive field. We will then specialize to the case of AdS_5 backgrounds corresponding to IR fixed points.

The ansatz (2.4) should cover more general cases of massive deformations dual to G_3 fields which lead to non conformal theories in the IR. The dual backgrounds of massive

⁴Actually, if we break the SU(2) symmetry, we can find a larger manifold of fixed points.

$\mathcal{N} = 4$ SYM are typically given by D3 branes dielectrically expanded into a D5 branes via Myers effect [14]. The supersymmetry parameter should satisfy a projection similar to (4.4), where the first term can be interpreted as the standard D3 brane projection and the second as a D5 brane wrapping the x, y directions in the internal manifold.

4.1 A family of solutions of the supersymmetric conditions

In this section, we review and generalize a class of supersymmetric solutions of the equations of motion of type IIB supergravity found by the group in USC [8]. This class of solutions is particularly suited for the description of massive deformations of conformal theories. The original solutions [8] were obtained for metrics with $U(1)^3$ isometries; here we will extend them to more general metrics so that they may be applied also to deformations of conformal theories associated with non toric Calabi-Yau manifolds. We describe complex solutions of the equations (3.4)-(3.10) that have at least one $U(1)$ isometry corresponding to the gauge theory R-symmetry and a constant dilaton corresponding to an exactly marginal direction of the conformal theory. The only non-zero fluxes are the RR 5-form and the complex 3-form G_3 .

We will work with the rescaled six-dimensional metric we introduced in the previous section (here the dilaton is set to zero)

$$ds_6^2 = e^{-2A} d\tilde{s}_6^2, \tag{4.7}$$

so that all the quantities in eqs (3.4)-(3.10) are defined with respect to the metric $d\tilde{s}_6^2$. The equations for the geometrical data read

$$d\left(e^{2A} e^{i(\alpha+\beta)} \sin 2\phi z\right) = 0, \tag{4.8}$$

$$d\left[j + \frac{i}{2} \cos 2\phi z \wedge \bar{z}\right] = 0. \tag{4.9}$$

We have seen that there is a preferred complex direction in the internal manifold specified by the conformally closed vector z . It is then natural to assume a four times two-dimensional splitting of the internal manifold

$$ds_6^2 = \eta_i A_{ij} \bar{\eta}_j + a_3 \eta_3 \bar{\eta}_3, \tag{4.10}$$

where the vielbein η_3 is proportional to z

$$z = \sqrt{a_3} \eta_3, \tag{4.11}$$

and the matrix A is hermitian

$$A = \begin{pmatrix} a_1 & a_0 \\ \bar{a}_0 & a_2 \end{pmatrix}. \tag{4.12}$$

For the vielbeins we choose the following ansatz

$$\begin{aligned} \eta_1 &= dz_1 + \alpha_1 dz_3 \\ \eta_2 &= dz_2 + \alpha_2 dz_3 \\ \eta_3 &= u dz_3. \end{aligned} \tag{4.13}$$

In the above expression z_i are local complex coordinates ($z_{1,2} = h_{1,2} + i\phi_{1,2}$, $z_3 = \ln u + i\phi_3$) and α_i are complex functions.

In terms of the above vielbeins the 2-forms defining the $SU(2)$ structure can be written as ⁵

$$j = \frac{i}{2} A_{ij} \eta_i \wedge \bar{\eta}_j, \tag{4.14}$$

$$\omega = i\sqrt{\det A} \eta_1 \wedge \eta_2. \tag{4.15}$$

The form of the z vector is determined by the massive deformation. Suppose we are adding the superpotential $W = \Phi_3^2$ for the adjoint field Φ_3 to a conformal field theory, for example $\mathcal{N} = 4$ SYM. We identify the complex coordinate $e^{z_3} = ue^{i\phi_3}$ with Φ_3 in the supergravity solution. From equation (2.22) we find

$$e^{2A} e^{i(\alpha+\beta)} \sin 2\phi z \sim dW = d(u^2 e^{2i\phi_3}), \tag{4.16}$$

so that

$$\begin{aligned} e^{2A} \sqrt{a_3} \sin 2\phi &= mu, \\ e^{i(\alpha+\beta)} &= e^{2i\phi_3}, \end{aligned} \tag{4.17}$$

where m is a constant. This automatically solves equation (4.8).

From eq. (4.9) it follows that there exists a closed 2-form

$$\begin{aligned} \hat{J} &= j + \frac{i}{2} \cos 2\phi z \wedge \bar{z} \\ &= \frac{i}{2} A_{ij} \eta_i \wedge \bar{\eta}_j + \frac{i}{2} \cos 2\phi a_3 \eta_3 \wedge \bar{\eta}_3. \end{aligned} \tag{4.18}$$

Therefore, although the metric is not Kähler, we can introduce, at least locally, a generalized Kähler potential F

$$\hat{J} = \frac{i}{2} \frac{\partial^2 F}{\partial z_i \partial \bar{z}_j} dz_i d\bar{z}_j. \tag{4.19}$$

Comparing (4.19) and (4.18), it is possible to express the functions a_i in the metric ansatz in terms of the generalized Kähler potential

$$A_{ij} = \frac{\partial^2 F}{\partial z_i \partial \bar{z}_j} \quad i, j = 1, 2, \tag{4.20}$$

$$A_{ij} \bar{\alpha}_j = \frac{\partial^2 F}{\partial z_i \partial \bar{z}_3}, \tag{4.21}$$

$$\alpha_i A_{ij} = \frac{\partial^2 F}{\partial \bar{z}_i \partial z_3}, \tag{4.22}$$

$$u^2 a_3 \cos 2\phi + \alpha_i A_{ij} \bar{\alpha}_j = \frac{\partial^2 F}{\partial z_3 \partial \bar{z}_3}. \tag{4.23}$$

⁵It is also possible to introduce another set of vielbeins that make the metric diagonal $X_1 = \sqrt{a_1} \eta_1 + \frac{a_0}{\sqrt{a_1}} \eta_2$, $X_2 = \frac{\sqrt{\det A}}{\sqrt{a_1}} \eta_2$ and $X_3 = \eta_3$. The defining two-forms become $j = \frac{i}{2} (X_1 \wedge \bar{X}_1 + X_2 \wedge \bar{X}_2)$ and $\omega = iX_1 \wedge X_2$.

It remains now to solve the equations involving the fluxes

$$e^{-4A} d\left(e^{4A} \cos 2\phi\right) = - * F_5, \quad (4.24)$$

$$e^{-4A} d\left(e^{2A} \sin 2\phi \operatorname{Im} \hat{\omega}\right) = \cos 2\phi H - * F_3, \quad (4.25)$$

$$\left[d\left(\frac{e^{-2A}}{\sin 2\phi} (\cos^2 \phi \hat{\omega} + \sin^2 \phi \bar{\hat{\omega}})\right) + iH \right] \wedge z = 0, \quad (4.26)$$

$$d\left(e^{-2A} \sin 2\phi \operatorname{Im} \hat{\omega} \wedge z \wedge \bar{z}\right) + 2iH \wedge \left(j + \frac{i}{2} \cos 2\phi z \wedge \bar{z}\right) = 0, \quad (4.27)$$

$$\frac{1}{2} e^{-2A} d\left(\cos 2\phi j^2 + ij \wedge z \wedge \bar{z}\right) + \sin 2\phi H \wedge \operatorname{Im} \hat{\omega} = 0. \quad (4.28)$$

The first of the equations involving fluxes, (4.24), can be considered as a definition of the five-form flux. To solve the other ones we introduce an ansatz for the complex three-form which is a straightforward generalization of that in [8]

$$G_3 = dA_2, \quad (4.29)$$

with $A_2 = C_2 - iB_2$ and

$$A_2 = \frac{2i}{m} e^{z_1+z_2-\bar{z}_3} [dz_1 \wedge dz_2 - \sin^2 \phi \eta_1 \wedge \eta_2]. \quad (4.30)$$

The above choice solves (4.26) with the condition

$$e^{z_1+z_2-\bar{z}_3} = m e^{i(\alpha-\beta)} \frac{\sqrt{\det A}}{e^{2A} \sin 2\phi}, \quad (4.31)$$

or, expanding the coordinates in real and imaginary part $z_i = h_i + i\phi_i$,

$$e^{h_1+h_2} = m \sqrt{\det A} a_3, \quad (4.32)$$

$$e^{i(\alpha-\beta)} = e^{i(\phi_1+\phi_2+\phi_3)}. \quad (4.33)$$

The meaning of this condition is that

$$\hat{\omega} \wedge z = \frac{i}{m} e^{z_1+z_2+z_3} dz_1 \wedge dz_2 \wedge dz_3, \quad (4.34)$$

so that $SU(3)$ structure holomorphic three-form $\Omega = \omega \wedge z$ is closed, up to a phase which can be reabsorbed in the definition of the vielbeins η_i .

From (4.27) one obtains

$$\begin{cases} \frac{\partial}{\partial \bar{z}_i} (a_3 u^2 \sin^2 \phi) = -\alpha_j A_{ji}, \\ \frac{\partial}{\partial z_i} (a_3 u^2 \sin^2 \phi) = -\bar{\alpha}_j \bar{A}_{ji} = -A_{ij} \bar{\alpha}_j, \end{cases} \quad (4.35)$$

which can be further simplified, using (4.21) and (4.22), to give

$$\begin{cases} \frac{\partial}{\partial \bar{z}_i} (a_3 u^2 \sin^2 \phi + \frac{\partial}{\partial z_3} F) = 0, \\ \frac{\partial}{\partial z_i} (a_3 u^2 \sin^2 \phi + \frac{\partial}{\partial \bar{z}_3} F) = 0. \end{cases} \quad (4.36)$$

Finally (4.28) gives

$$\frac{\partial}{\partial z_3}(a_3 u^2 \sin^2 \phi + \frac{\partial}{\partial \bar{z}_3} F) = 0. \quad (4.37)$$

For backgrounds that do not depend on ϕ_3 we can solve all the previous equations by taking

$$a_3 u^2 \sin^2 \phi = -\frac{\partial}{\partial z_3} F. \quad (4.38)$$

At this point a long but straightforward computation shows that equation (4.25) is also satisfied.

Notice that the solution is completely specified by a generalized Kähler potential $F(z_i)$ that satisfies the condition of closure for Ω , equation (4.32). Indeed, the metric is determined by F , the fluxes by equation (4.24) and by the ansatz (4.29), ϕ by condition (4.38), the warp factor A by equation (4.17) and, finally, the two phases α and β by (4.17)–(4.33). The situation is similar to the Calabi-Yau case where the solution is determined by a Kähler potential which satisfies the condition of closure for Ω . We see that, in this particular class of solutions, the inclusion of fluxes does not introduce new constraints.

The equations further simplify if the metric has three $U(1)$ isometries. This is the case where we introduce a mass deformation in $\mathcal{N} = 4$ SYM or more generally in a theory dual to a toric Calabi-Yau. The mass term typically breaks one global symmetry of the theory, but in the supergravity solution this breaking can be included in the phase of the G field and does not affect the metric. We can then have a solution where $F(h_i)$ does not depend on the angles ϕ_i , the metric has a $U(1)^3$ isometry and the G field has a phase $e^{i(\phi_1+\phi_2+\phi_3)}$ according to equation (4.29). In this toric cases, \hat{J} can be rewritten as

$$\hat{J} = \frac{1}{4} \frac{\partial^2 F}{\partial h_i \partial h_j} dh_i \wedge d\phi_j = d \left(\frac{1}{4} \frac{\partial F}{\partial h_i} d\phi_i \right). \quad (4.39)$$

Here we used $dz_i = dh_i + id\phi_i$, $h_3 = \log u$. The quantities $\frac{\partial F}{\partial h_i}$ then play the role of momentum map variables for the $U(1)^3$ fibration. The PW flow belongs to this toric class of solutions.

4.2 AdS_5 solutions

We obtain AdS_5 solutions by restricting to warped conical six-dimensional metrics,

$$d\tilde{s}_6^2 = H^4 dr^2 + r^2 ds_5^2(y) \quad (4.40)$$

where H only depends on the angular coordinates y_i of the base ds_5 . In this way the full 10-dimensional metric

$$ds_{10}^2 = e^{2A} ds_4^2 + e^{-2A} [H^4 dr^2 + r^2 ds_5^2] \quad (4.41)$$

factorizes to the warped AdS form

$$H^2 ds_{AdS}^2 + H^{-2} ds_5^2, \quad (4.42)$$

provided that

$$e^{2A} = r^2 H^2. \quad (4.43)$$

Conformal invariance requires $F \sim r^2$, $A_{ij} \sim r^2$ and α_i adimensional. The correct scaling behavior of the metric combined with equation (4.17) also requires $u \sim r^{3/2}$ and $a_3 \sim 1/r$. This is a consequence of the marginality of the superpotential term Φ_3^2 at the fixed point, which in turn implies dimension 3/2 for the field Φ_3 ⁶ associated with the variable u . It is known that, for a Kähler cone, $F = 4r^2$ [22]. In our case, more generally,

$$F = r^2 f(y), \quad (4.44)$$

where f is a function on the five-dimensional base. The conditions for the existence of an AdS_5 solution finally require the absence of terms linear in dr in the six-dimensional metric. This represents a further constraint on the coordinates z_i and the Kähler potential F .

All these conditions considerably simplify in the toric case. We can choose coordinates λ, k, ϕ_i on the five-dimensional base such that

$$\begin{aligned} dz_i &= dh_i + id\phi_i = n_i \frac{dr}{r} + dv_i(\lambda, k) + id\phi_i, & i = 1, 2 \\ dz_3 &= \frac{du}{u} + id\phi_3 = n_3 \frac{dr}{r} + \frac{d\lambda}{\lambda} + d\phi_3, & n_3 = \frac{3}{2}. \end{aligned} \quad (4.45)$$

The generalized Kähler potential only depends on h_i , $F(h_i)$. We require $n_1 + n_2 + n_3 = 3$ in order for the (3,0) holomorphic form Ω to scale as r^3 (cfr equation (4.34)).

Determining the metric from $J = \hat{J} + (1 - \cos 2\phi)iz\bar{z}/2$

$$ds_6^2 = \frac{1}{4} \frac{\partial^2 F}{\partial h_i \partial h_j} dz_i d\bar{z}_j + (1 - \cos 2\phi) a_3 u^2 dz_3 d\bar{z}_3, \quad (4.46)$$

and using repeatedly the condition of scale invariance

$$n_i \frac{\partial F}{\partial h_i} = r \frac{\partial F}{\partial r} = 2F, \quad (4.47)$$

we obtain the condition for the absence of mixed terms in dr

$$dF = 2F \frac{dr}{r} + 3u \frac{\partial F}{\partial u} \frac{d\lambda}{\lambda}. \quad (4.48)$$

This implies that

$$F = r^2 f(\lambda) \quad (4.49)$$

and

$$\lambda f'(\lambda) = \frac{3}{r^2} u \frac{\partial F}{\partial u}. \quad (4.50)$$

Finally, the warp factor is determined by the dr^2 term in the metric which, with simple manipulations, can be written as

$$H^4 = f(\lambda) - \frac{3}{4} \lambda f'(\lambda). \quad (4.51)$$

⁶This is better thought as the composite $[\Phi_1, \Phi_2]$ at the IR fixed point.

As an example we can reconstruct the PW solution for the IR fixed point of massive $\mathcal{N} = 4$ SYM. The undeformed solution is associated with \mathbb{C}^3 for which we choose coordinates

$$\begin{aligned} e^{z_1} &= r \cos \theta \cos \varphi e^{i\phi_1}, \\ e^{z_2} &= r \cos \theta \sin \varphi e^{i\phi_2}, \\ e^{z_3} &= r \sin \theta e^{i\phi_3}. \end{aligned} \tag{4.52}$$

The complex coordinates for the IR fixed point are just obtained with a rescaling

$$\begin{aligned} e^{z_1} &= r^{3/4} \cos \theta \cos \varphi e^{i\phi_1}, \\ e^{z_2} &= r^{3/4} \cos \theta \sin \varphi e^{i\phi_2}, \\ e^{z_3} &= r^{3/2} \sin \theta e^{i\phi_3}. \end{aligned} \tag{4.53}$$

We can choose $\lambda = \sin \theta$ and $k = \varphi$. The solution of the constraint (4.50) is obtained by taking

$$F = cr^2(1 - 2\sin^2 \theta), \tag{4.54}$$

from which $H^4 = c(1 + \sin^2 \theta)$. All other quantities are consistently determined by the equations in the previous section. We obtain for example

$$\cos^2 \phi = \frac{\sin^2 \theta}{2(1 + \sin^2 \theta)}. \tag{4.55}$$

The full PW metric can be reconstructed as

$$\begin{aligned} d\tilde{s}_6^2 &= \frac{3}{4}dr^2(1 + \sin^2 \theta) + r^2 \left[\frac{1}{2}(1 + \sin^2 \theta)d\theta^2 + \cos^2 \theta d\varphi^2 + \cos^2 \theta (\cos^2 \phi d\phi_1^2 + \sin^2 \phi d\phi_2^2) \right. \\ &\quad \left. + \frac{\sin^2 \theta}{2}d\phi_3^2 + \frac{(\cos^2 \theta (\cos^2 \phi d\phi_1 + \sin^2 \phi d\phi_2) + \sin^2 \theta d\phi_3)^2}{3(1 + \sin^2 \theta)} \right] \end{aligned} \tag{4.56}$$

which is equivalent to formula (7.8) of the second paper in [8].

The moduli space of vacua of the dual gauge theory is two-dimensional and spanned by the vacuum expectation values of the fields $\Phi_{1,2}$. The moduli space for D3-brane probes of the PW solution is obtained by imposing equation (3.14)

$$\sin 2\phi = \frac{\sin \theta \sqrt{2 + \sin^2 \theta}}{1 + \sin^2 \theta} \equiv 0, \tag{4.57}$$

which selects the two-dimensional sub-manifold $u = r^{3/2} \sin \theta \equiv 0$. On this sub-manifold $\cos 2\phi \equiv 1$ and

$$\hat{J} \equiv J, \tag{4.58}$$

so that the moduli space seen by the probe is a two-dimensional Kähler manifold as required by supersymmetry for $\mathcal{N} = 1$ gauge theories ⁷.

⁷An explicit computation of the Kähler potential for a probe in the PW flow was performed in [16].

It would be interesting to look for other warped AdS_5 solutions of type IIB supergravity. The above analysis has been performed for massive deformations of $\mathcal{N} = 4$ Super Yang-Mills, thus for deformation of $AdS_5 \times S^5$. However it is likely to describe also mass deformations of $\mathcal{N} = 1$ quiver gauge theories associated with more general Sasaki-Einstein manifolds H and backgrounds of the form $AdS_5 \times H$. In quiver gauge theories, the addition of a mass term for a single adjoint field leads quite generically to an IR fixed point if there are enough global abelian symmetries⁸. This is typically the case for Sasaki-Einstein backgrounds where massless vectors arise not only from isometries but also from RR fields. We thus expect a large number of AdS_5 solutions dual to massive deformations of conformal theories. Some of them will be still described by Sasaki-Einstein backgrounds⁹ but others will be described by warped solutions with non zero G flux. At the moment, the only explicitly known example in the latter class is PW and it would be quite interesting to find alternative ones. The number of known Sasaki-Einstein metrics has been recently enlarged with the discovery of the Y^{pq} and L^{pqr} metrics [17]. In particular the quivers associated with the Generalized Conifolds, which are particular cases of the L^{pqr} [18], have adjoint fields and massive deformations leading to IR fixed points, which could correspond to warped AdS_5 solutions with fluxes.

5. Marginal deformations of conformal field theories

A second interesting class of $SU(2)$ structure backgrounds is provided by the duals of marginal deformations of conformal gauge theories.

Once again we consider the example of $\mathcal{N} = 4$ SYM. It is known that there exists a manifold of $\mathcal{N} = 1$ fixed points that contains the $\mathcal{N} = 4$ Yang-Mills theory [13]. The corresponding theories can be described in $\mathcal{N} = 1$ language as containing the same fields as $\mathcal{N} = 4$ SYM but with a superpotential (modulo the $SU(3)$ global symmetry)

$$h \text{Tr}(e^{i\pi\beta}\Phi_1\Phi_2\Phi_3 - e^{-i\pi\beta}\Phi_1\Phi_3\Phi_2) + h' \text{Tr}(\Phi_1^3 + \Phi_2^3 + \Phi_3^3). \quad (5.1)$$

In addition to the three parameters appearing in the superpotential, we have the complexified coupling constant. There is a particular relation between the four complex parameters g_{YM}, h, h', β for which the theory is superconformal [13]. The equations for the vanishing of all beta-functions are satisfied if

$$\Delta_{\Phi_i}(g_{YM}, h, h', \beta) = 1. \quad (5.2)$$

⁸The conditions for conformal invariance of the original theories are determined only up to free parameters associated with the abelian symmetries, since the latter can mix with the R-symmetry. The exact dimensions of the chiral fields is determined using a-maximization [15]. Upon the addition of a mass term, we should just restrict the maximization on the sub-variety where the R-charge of the massive field is one. We thus expect a solution of the a-maximization if the original theory have enough abelian symmetries.

⁹Not all the mass deformations are dual to modes of the three-form G field in supergravity, as it happens for $\mathcal{N} = 4$ SYM. For a generic quiver theory, where H may have orbifold singularities, some mass deformations are dual to geometrical blowing up modes. A familiar example is provided by the $\mathcal{N} = 2$ quiver gauge theory associated with the singular Calabi-Yau $\mathbb{C}^2/Z_2 \times \mathbb{C}$, i.e. $H = S^5/Z_2$. A mass deformation dual to the blow up mode leads to an IR fixed point corresponding to the conifold. In this cases, the IR fixed point will be still of the form $AdS_5 \times H$ with H a Sasaki-Einstein manifold.

Using the obvious permutation symmetry among the Φ_i we conclude that we have a single equation for four unknowns. This yields a three-dimensional complex manifold of fixed points.

We consider here the case of the β -deformation with $h' = 0$,

$$h \operatorname{Tr}(e^{i\pi\beta}\Phi_1\Phi_2\Phi_3 - e^{-i\pi\beta}\Phi_1\Phi_3\Phi_2), \quad (5.3)$$

whose supergravity dual has been found in [7] and it will be referred as the LM solution. The theory still preserves a $U(1) \times U(1)$ global symmetry, in addition to the R-symmetry. The global $U(1) \times U(1)$ acts with charges $(0, 1, -1)$ and $(-1, 1, 0)$ on the three chiral fields Φ_i . The deformation of the superpotential modifies the F-term equations and reduces the moduli space of vacua of the theory. The F-terms read

$$\Phi_1\Phi_2 = e^{-2\pi i\beta}\Phi_2\Phi_1, \quad \Phi_3\Phi_1 = e^{-2\pi i\beta}\Phi_1\Phi_3, \quad \Phi_2\Phi_3 = e^{-2\pi i\beta}\Phi_3\Phi_2. \quad (5.4)$$

The original moduli space of $\mathcal{N} = 4$ SYM was parametrized by arbitrary diagonal matrices Φ_i . In the β -deformed theory, we see that a diagonal matrix for, say, Φ_i solves the F-term equations only if the other two Φ_k vanish. The mesonic moduli space is then the union of three branches meeting at the origin. A D3-probe sees a three-dimensional moduli space isomorphic to \mathbb{C}^3 in the case of $\mathcal{N} = 4$ SYM but only three complex lines intersecting at the origin in the case of the β -deformed theory. It is known [19] that, for special values of the deformation parameter where β is rational, we can have other Coulomb branch vacua where Φ_i define a non-commutative torus. As usual, we do not consider in this paper baryonic type vacua.

We will further restrict to the case where β is real. The case where β is complex can be obtained by a further type IIB S duality. The case of real β is particularly interesting because it can be obtained from the $\mathcal{N} = 4$ solution by a T-duality. The T-duality transformation acts on the two-torus made with two $U(1) \times U(1)$ isometries of the original solution. As we will show in the following, a T-duality on two angular directions corresponding to isometries of \mathbb{C}^3 , or more generally of a Calabi-Yau, transforms the original $SU(3)$ structure into an $SU(2)$ structure satisfying the special ansatz (3.1), thus of the special form considered in this paper.

5.1 Spinors and the action of T-duality

The generating solution technique used in [7] applies to all backgrounds with at least two isometries. We call $\varphi_{1,2}$ the two corresponding angles and $g_{ij}, B_{ij} \equiv B\epsilon_{ij}$ ($i, j = 1, 2$) the metric and the antisymmetric NS two-form on the T^2 spanned by $\varphi_{1,2}$. The metric thus reads

$$ds_6^2 = g_{ij}e_{\varphi_i}e_{\varphi_j} + d\tilde{s}_4^2 = y_1^2 + y_2^2 + d\tilde{s}_4^2, \quad (5.5)$$

where the one-forms $e_{\varphi_i} = d\varphi_i + \dots$ have been used to eliminate off-diagonal terms, if needed, and $d\tilde{s}_4^2$ does not depend on φ_i . $y_{1,2}$ correspond to a choice of vielbeins along the T^2 .

In order to obtain new solutions, we can use the $SL(2, \mathbb{R})$ subgroup of the T-duality group $O(2, 2)$ that acts on the complexified Kähler modulus $\nu = B + i\sqrt{g}$ of the two torus as

$$\nu \rightarrow \frac{a\nu + b}{c\nu + d}. \quad (5.6)$$

As argued in [7], the particular element

$$LM = \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix} \quad (5.7)$$

transforms regular solutions in other completely regular solutions depending on the parameter γ . Starting with a Calabi-Yau background with two isometries and no B -field, we obtain a new background with $\sqrt{g'} = G\sqrt{g}$ and $B' = \gamma g G$ where $G = 1/(1 + \gamma^2 g)$. The T-dual solution is then

$$ds_6^2 = G g_{ij} e_{\varphi_i} e_{\varphi_j} + d\tilde{s}_4^2 = G(y_1^2 + y_2^2) + d\tilde{s}_4^2. \quad (5.8)$$

Formulae for the explicit action of the T-duality group on various quantities are collected in appendix B. We need in particular the action of T-duality on the supersymmetry spinors. Using the formulae given in the appendix, we can show that

$$\eta_+^{1'} = \eta_+^1, \quad (5.9)$$

$$\eta_+^{2'} = \Omega_T \eta_+^2, \quad (5.10)$$

where

$$\Omega_T = \frac{1}{\sqrt{1 + \gamma^2 g}} (1 + \gamma \Gamma_{\varphi_1 \varphi_2}). \quad (5.11)$$

If η_+^0 is the $U(1) \times U(1)$ invariant spinor of the original CY, after T-duality we have

$$\eta_+^1 = \eta_+^0, \quad \eta_+^2 = -i \frac{1}{\sqrt{1 + \gamma^2 g}} \left(\eta_+^0 + \gamma \sqrt{g} \frac{z}{2} \eta_-^0 \right), \quad (5.12)$$

where, similarly to section 4, we have rewritten the action of the gamma-matrices in terms of a normalized vector z

$$\Gamma_{\varphi_1 \varphi_2} \eta_+^0 = \sqrt{g} \Gamma_{y_1 y_2} \eta_+^0 = \sqrt{g} \frac{z}{2} \eta_-^0. \quad (5.13)$$

The spinors (5.12) satisfy condition (3.13) and thus can be brought to the form (3.1)

$$\eta_+^1 = a\eta_+ + b \frac{z}{2} \eta_-, \quad \eta_+^2 = -i(a\eta_+ - b \frac{z}{2} \eta_-), \quad (5.14)$$

with

$$\begin{aligned} \eta_+ &= \frac{\bar{a}\eta_+^0 - b(z/2)\eta_-^0}{|\eta_+^0|^2}, \\ b &= \frac{-\gamma\sqrt{g}}{1 + \sqrt{1 + \gamma^2 g}} \bar{a}, \\ |a|^2 &= \frac{1 + \sqrt{1 + \gamma^2 g}}{2\sqrt{1 + \gamma^2 g}} |\eta_+^0|^2, \end{aligned} \quad (5.15)$$

where $|\eta_+^0|^2 = e^A$. This transformation is just a rotation in the U(1) family of almost complex structures allowed by the SU(2) structure. By comparison with the ansatz (3.2), we can also compute

$$\begin{aligned} \sin 2\phi &= -\frac{\gamma\sqrt{g}}{\sqrt{1+\gamma^2g}}, \\ \cos 2\phi &= \frac{1}{\sqrt{1+\gamma^2g}}. \end{aligned} \tag{5.16}$$

We want now to construct the pure spinors after the action of T-duality. These can be easily obtained from (2.13) by tensoring the transformed spinors (5.12). It is however possible to compute directly the action of T-duality on the pure spinors using the results of [20]. This is because, via Clifford map, pure spinors can be thought as bispinors. Specializing the action of T-duality on bispinors as given in [20] we have

$$T : \quad \Phi_{\pm} \rightarrow \Phi_{\pm}\Omega_T^{\dagger}, \tag{5.17}$$

where Ω_T is given in (5.11) and it acts on the pure spinors by Clifford multiplication from the right.

As the T-duality was applied to flat space (or more generally to a CY) we should start from the standard pair of pure spinors (Φ_+, Φ_-) corresponding to SU(3) structure, namely the exponentiated fundamental form and the holomorphic three-form. In our notations, these are obtained (modulo normalization) by taking $a = 1, x = -i$ and $b = y = 0$. In a background with two isometries, if we write the metric in diagonal form

$$ds_6^2 = x_1^2 + x_2^2 + y_1^2 + y_2^2 + z\bar{z}, \tag{5.18}$$

where z is the normalized vector introduced in (5.13) and y_i correspond to the fiber directions along which T-duality acts, Φ_{\pm} read

$$\begin{aligned} \Phi_+ &= \frac{i}{8}e^{-iJ} = \frac{i}{8}e^{z\bar{z}/2}e^{-ij}, \\ \Phi_- &= -\frac{1}{8}z \wedge \omega, \end{aligned} \tag{5.19}$$

with ¹⁰

$$\begin{aligned} j &= x_1 \wedge y_1 + x_2 \wedge y_2, \\ \omega &= i(x_1 + iy_1) \wedge (x_2 + iy_2). \end{aligned} \tag{5.20}$$

Note that, even though the physical context is different, the action of the second piece in Ω_T , which essentially amounts to taking the Hodge star along the fibers, is exactly the same as for the maximally type-changing T-duality action discussed in [21], which exchanges e^{-ij} and ω . Indeed, on the forms written above it is not hard to verify that $(e^{-ij})\Gamma_{y_1y_2}^{\dagger} = -i\omega$

¹⁰In determining the phase in ω we used definition (2.5) and the fact that $\chi_0 = z\eta_0^*/2 = \Gamma_{y_1y_2}\eta_0^*$.

and $(\omega)\Gamma_{y_1 y_2}^\dagger = -ie^{-ij}$, so that we obtain ¹¹

$$\begin{aligned}\Phi_+ \Omega_T^\dagger &= \frac{i}{8\sqrt{1+\gamma^2 g}} e^{z\bar{z}/2} \left[e^{-ij} - i\gamma\sqrt{g}\omega \right], \\ \Phi_- \Omega_T^\dagger &= \frac{-1}{8\sqrt{1+\gamma^2 g}} z \wedge \left[\omega - i\gamma\sqrt{g}e^{-ij} \right].\end{aligned}\tag{5.21}$$

Finally we can bring the pure spinors to the general form that has been used throughout the paper by making use of the connection between the rotation of the four-dimensional complex structure and the γ parameter given in (5.15) and (5.16). This amounts to a redefinition of the real two-forms

$$\begin{pmatrix} \cos 2\phi & -\sin 2\phi \\ \sin 2\phi & \cos 2\phi \end{pmatrix} \begin{pmatrix} j \\ \text{Re } \omega \end{pmatrix} \longrightarrow \begin{pmatrix} j \\ \text{Re } \omega \end{pmatrix}.\tag{5.22}$$

The phases α and β are set to zero here, so we are dropping the hats on ω .

In principle we could have just stopped this discussion here, but before proceeding to the discussion of the $SU(2)$ structure solution, we would like to present an interpretation of the LM transformation (5.7) in terms of Generalized Complex Geometry. The idea is to describe the change of type of the pure spinors given by (5.21) in terms of the standard $O(2,2)$ action on $TM \oplus T^*M$ (where M is the T^2 along which we T-dualize) ¹². As already mentioned, the nontrivial part of the Ω_T action is given by the Hodge star along the fibers which in turn is conveniently captured by the Clifford action of $\Gamma_{y_1 y_2}$. When acting on the forms (pure spinors) the product of gamma-matrices gives terms with full antisymmetrization, full contraction and partial contraction. Their three $O(2,2)$ counterparts are [11]:

- B -transform - a shear transformation on the cotangent bundle T^*M which acts by wedging the forms with an exponentiated two form and does not change the type of the pure spinors;
- β transform - a shear transformation on the tangent bundle TM which acts by contracting the forms with an exponentiated bivector and does change the type of the pure spinors;
- $SL(2)$ rotation A - a vector-valued one-form which acts by a contraction and a wedge. The form of these operators is particularly simple in two dimensions:

$$B = y_1 \wedge y_2, \quad \beta = \iota_{y_1} \wedge \iota_{y_2}, \quad A = -(y_1 \wedge \iota_{y_2} - y_2 \wedge \iota_{y_1}),\tag{5.23}$$

¹¹We can understand the general action of $\Gamma_{y_1 y_2}$ on e^{-ij} and ω without referring to a particular basis as done above. We may use the fact that these forms are self-dual when restricted to the four dimensions transverse to the directions spanned by the complex vector z . Moreover, their respective real and imaginary parts form a full basis of self-dual forms: $(1 - j^2/2)$, j , $\text{Re } \omega$ and $\text{Im } \omega$. The star in the fiber directions only (i.e. the action of Γ_{12}) is a map among self-dual forms but mixes forms of different rank. So the transformed pure spinor will simply be a combination of $(1 - j^2/2)$, j , $\text{Re } \omega$ and $\text{Im } \omega$ as in (2.13).

¹²In principle GCG (or simply geometrical) descriptions can be problematic when a B -field with two legs along the fibers is involved, as it is the case here. However since the latter is generated by $SL(2)$ rotations and is constant along the fibers we will see that the geometrical description is perfectly adequate here.

where ι denotes a contraction. Their actions on the pure spinors are as follows (we ignore the z contribution which is inert under these actions):

$$B(e^{-ij}) = y_1 \wedge y_2, \quad \beta(e^{-ij}) = -x_1 \wedge x_2, \quad A(e^{-ij}) = i(x_2 \wedge y_1 - x_1 \wedge y_2), \quad (5.24)$$

$$B(\omega) = ix_1 \wedge x_2 \wedge y_1 \wedge y_2, \quad \beta(\omega) = i, \quad A(\omega) = x_1 \wedge y_1 + x_2 \wedge y_2. \quad (5.25)$$

One can define the combined action of these operations on the pure spinors (5.19) as

$$e^{\hat{B}}\Phi_{\pm} := e^{2\phi(B+\beta+A)}\Phi_{\pm}.$$

Using that $A(B + \beta) = (B + \beta)A = 0$ on all the forms and $(B + \beta + A)^2\Phi_{\pm} = -\Phi_{\pm}$, we find

$$e^{\hat{B}}\Phi_{\pm} = [\cos 2\phi + \sin 2\phi(B + \beta + A)]\Phi_{\pm}. \quad (5.26)$$

Using the explicit actions (5.24) and (5.25) on Φ_+ and Φ_- , we recover the form (5.21) for the transformed pure spinors provided that we identify the parameter ϕ as in (5.16). Note that Ω_T does not break into series of B -transforms, β -transforms (these two do not commute!) and $SL(2)$ rotations. Instead its action is given by a Clifford multiplication on pure spinors which can be seen as a type-changing “generalized \hat{B} -transform”. While the decomposition (5.26) was proved using a specific basis for j and ω , it is expected to hold more generally. It would be curious to see if a similar decomposition may hold for (at least a class of) higher-dimensional T-dualities.

5.2 The geometrical structure of the LM solution

In this section we analyze the geometry of the LM solution. We use the rescaled six-dimensional metric $ds_6^2 = e^{-2A}d\tilde{s}_6^2$.

The original Calabi-Yau cone for $\mathcal{N} = 4$ SYM is just \mathbb{C}^3 . We choose three complex coordinates $z_i = r\mu_i e^{i\phi_i}$ representing the three adjoint scalar fields Φ_i . A convenient parametrization for the z_i is

$$\begin{aligned} z_1 &= r \cos \alpha e^{i(\psi - \varphi_2)}, \\ z_2 &= r \sin \alpha \cos \theta e^{i(\psi + \varphi_1 + \varphi_2)}, \\ z_3 &= r \sin \alpha \sin \theta e^{i(\psi - \varphi_1)}. \end{aligned} \quad (5.27)$$

In this parametrization, the two $U(1)$ symmetries act by shifting φ_1 and φ_2 , respectively.

The form of the vector z can be determined by equation (5.13). By expressing the action of $\Gamma_{\varphi_1\varphi_2}$ in the complex basis given above we find

$$z \sim dZ, \quad Z = z_1 z_2 z_3 = r^3 \mu_1 \mu_2 \mu_3 e^{3i\psi}. \quad (5.28)$$

Notice that the complex structure on \mathbb{C}^3 given by the coordinates z_i is the one determined by the spinor η_+^0 . By definition, this means that it is the Clifford vacuum annihilated by the complexified gamma-matrices associated to z_i . A simple computation using the explicit parametrization (5.27) then gives the result quoted above. We can then adapt our

metric to the four plus two structure determined by the vector z by considering the following vielbeins

$$\begin{aligned}\chi_1 &= e^{-i\phi_1} \sqrt{\frac{g}{\mu_1^2(\mu_2^2 + \mu_3^2)}} \left(dz_1 - \frac{\bar{z}_2 \bar{z}_3 dZ}{r^4 g} \right) \equiv x_1 + iy_1, \\ \chi_2 &= e^{-i\phi_2} \sqrt{1 + \frac{\mu_3^2}{\mu_2^2}} \left(dz_2 - \frac{\bar{z}_1 \bar{z}_3 dZ}{r^4 g} \right) + \frac{\mu_3^2 e^{-i\phi_1}}{\mu_1 \sqrt{\mu_2^2 + \mu_3^2}} \left(dz_1 - \frac{\bar{z}_2 \bar{z}_3 dZ}{r^4 g} \right) \equiv x_2 + iy_2,\end{aligned}$$

where

$$g = \mu_1^2 \mu_2^2 + \mu_2^2 \mu_3^2 + \mu_3^2 \mu_1^2 = \sin^2 \alpha (\cos^2 \alpha + \sin^2 \alpha \sin^2 \theta \cos^2 \theta).$$

These combine with z to give an integrable complex structure on \mathbb{C}^3

$$ds_6^2 = \sum_{i=1}^3 dz_i d\bar{z}_i = \chi_1 \bar{\chi}_1 + \chi_2 \bar{\chi}_2 + \frac{dZ d\bar{Z}}{r^4 g} \equiv x_1^2 + x_2^2 + y_1^2 + y_2^2 + \frac{dZ d\bar{Z}}{r^4 g}. \quad (5.29)$$

The metric on T^2 is simply given by the terms $y_1^2 + y_2^2$. Explicitly

$$\begin{aligned}y_1 &= r \sqrt{\cos^2 \alpha + \sin^2 \alpha \sin^2 \theta \cos^2 \theta} \left(d\varphi_2 - \frac{\cos^2 \alpha - 2 \sin^2 \alpha \sin^2 \theta \cos^2 \theta}{\cos^2 \alpha + \sin^2 \alpha \cos^2 \theta \sin^2 \theta} d\psi \right), \\ y_2 &= r \sin \alpha (d\varphi_1 + \cos^2 \theta d\varphi_2 + \cos 2\theta d\psi).\end{aligned}$$

We now perform the T-duality transformation (5.7) to the 10 dimensional metric

$$ds^2 = r^2 ds_4^2 + \frac{1}{r^2} ds_6^2.$$

The original B -field is zero and the ν parameter of the two-torus is simply given by $\nu = i\sqrt{g}$. The six-dimensional internal metric after T-duality

$$ds^2 = r^2 ds_4^2 + \frac{1}{r^2} ds_{ML}^2$$

with its natural almost complex structure is now given by

$$\begin{aligned}ds_{LM}^2 &= \chi'_1 \bar{\chi}'_1 + \chi'_2 \bar{\chi}'_2 + \frac{dZ d\bar{Z}}{r^4 g} \equiv x_1^2 + x_2^2 + G(y_1^2 + y_2^2) + \frac{dZ d\bar{Z}}{r^4 g}, \\ \chi'_i &= x_i + i\sqrt{G} y_i.\end{aligned} \quad (5.30)$$

In real coordinates we can also write the metric as in [7]

$$ds_{LM}^2 = dr^2 + r^2 \left(\sum_{i=1}^3 (d\mu_i^2 + g\mu_i^2 d\phi_i^2) + 9\gamma^2 G \mu_1^2 \mu_2^2 \mu_3^2 d\psi^2 \right). \quad (5.31)$$

The expressions for the other fields can be easily found by using the rules for T-duality given in appendix B and read

$$\begin{aligned}e^{2A} &= r^2, \\ e^\phi &= \sqrt{G}, \\ B_2 &= \gamma \sqrt{g} G \frac{y_1 \wedge y_2}{r^2}, \\ F_3 &= 12\gamma \cos \alpha \sin^3 \alpha \sin \theta \cos \theta d\psi \wedge d\alpha \wedge d\theta, \\ F_5 &= 4(\text{volAdS}_5 + *\text{volAdS}_5).\end{aligned} \quad (5.32)$$

It is immediate to check that the vector

$$z = \frac{dZ}{r^2 \sqrt{g}} \tag{5.33}$$

is conformally closed (equation (3.4))

$$d\left(e^{2A-\varphi} e^{i(\alpha+\beta)} \sin 2\phi z\right) = 0 \tag{5.34}$$

with $\alpha = \beta = 0$. From equation (2.22) we also find

$$dW \sim e^{2A-\varphi} e^{i(\alpha+\beta)} \sin 2\phi z = d(-\gamma z_1 z_2 z_3) \tag{5.35}$$

which exactly corresponds to the (abelianized) superpotential $W \sim \Phi_1 \Phi_2 \Phi_3$ in (5.3).

The metric is not Kähler but the condition of supersymmetry (3.5) requires the existence of a conformally closed two form

$$d\left(e^{-\varphi} \hat{J}\right) = d\left[e^{-\varphi} \left(j + \frac{i}{2} \cos 2\phi z \wedge \bar{z}\right)\right] = 0. \tag{5.36}$$

This equation is also trivially satisfied since

$$\hat{J} = \frac{i}{2}(\chi'_1 \bar{\chi}'_1 + \chi'_2 \bar{\chi}'_2 + \sqrt{G} z \bar{z}) = \sqrt{G}(x_1 y_1 + x_2 y_2 + \frac{i}{2} z \bar{z}) = e^\varphi J_{N=4}. \tag{5.37}$$

It is then straightforward to check that all other conditions for supersymmetry (3.6)–(3.10) are satisfied with the definition (5.20). For this check we used the almost complex structure defined in equation (5.30). As discussed in details in the previous section, our supersymmetry conditions are satisfied with a specific choice of almost complex structure, which is obtained by applying a rotation (5.15)–(5.22) to the one defined by the spinor η_0 . An explicit computation shows that the result is indeed the almost complex structure defined in formula (5.30).

We can also analyze the moduli space for probe D3 branes in this background. The general formula (3.14) requires

$$\sin 2\phi = -\gamma \sqrt{G} \sqrt{g} \equiv 0, \tag{5.38}$$

or equivalently

$$g = \mu_1^2 \mu_2^2 + \mu_2^2 \mu_3^2 + \mu_3^2 \mu_1^2 \equiv 0, \tag{5.39}$$

which determines the three branches

$$z_i = z_j = 0, z_k \neq 0, \quad i \neq j \neq k \tag{5.40}$$

in agreement with the field theory analysis.

As already mentioned, for rational γ there are other Coulomb branch vacua. These have been identified in [7] as n_5 D5 branes wrapped on the two torus (in points where it is not vanishing) with $1/\gamma$ units of gauge flux. These objects carry both units of D5 charge (n_5) and units of D3 charge (n_5/γ) and satisfy the charge quantization condition only for

γ rational. We can easily check that these are BPS states using conditions (2.20). For a D5 brane wrapped on the two torus and the spinorial ansatz (3.1), the supersymmetry conditions become $\text{Re}\hat{J} = 0$ which is automatically satisfied and

$$\sin 2\phi(F - B) + \cos 2\phi G y_1 \wedge y_2 = 0 \tag{5.41}$$

which, given the form of the B field (5.32), is satisfied exactly by $F = y_1 \wedge y_2 / (\gamma\sqrt{g})$.

A marginal deformation analogous to the β -deformation exists for all quiver gauge theories associated with toric Calabi-Yau singularities. All these backgrounds have indeed two isometries commuting with the supersymmetry generators, or equivalently, from the field theory point of view, two U(1) global symmetries in addition to the R-symmetry. The generating technique of [7] then determines the supergravity dual of the marginally deformed conformal field theory. The above analysis of the geometrical structure of the supersymmetric solution will apply to all these backgrounds with minor changes.

6. Conclusions

In this paper we started a detailed analysis of the conditions of supersymmetry for SU(2) structure backgrounds. While our results are not the most general ones, since the solution relies on a particular choice in the spinorial ansatz (3.1), we were able to write a very simple set of equations for the SU(2) invariant forms and the fluxes for a very large class of backgrounds. In particular, generalizing results in [8], we described a simple class of complex manifolds and associated fluxes which solve the supersymmetry conditions of type IIB and are characterized by the existence of a generalized Kähler potential.

The use of G-structures has been useful in the past not only for characterizing known solutions but also for finding new ones. One example is the baryonic branch of the Klebanov-Strassler solution [3]. In this paper, we studied the geometry of the PW and LM solutions describing massive and marginal deformations of conformal theories. It would be quite interesting to pursue further the analysis of this paper and try to find new conformal and non conformal backgrounds. There are various obvious directions where one could move. For example, there should be a large number of conformal gauge theories corresponding to warped AdS_5 solutions with fluxes. We gave a possible characterization of those solutions corresponding to massive deformations of conformal gauge theories in section 4. PW is the only known solution and it would be interesting to find new examples. Moreover, there exist other marginal deformations of $\mathcal{N} = 4$ SYM and other conformal theories beside the β -deformation. In particular, it would be interesting to find the marginal deformation of $\mathcal{N} = 4$ associated with the coupling h' in equation (5.1).

We are only making the first steps understanding the interplay between the GCY geometry of the internal manifolds and that of moduli spaces of the dual gauge theories. Much of the information about the solution, including the type of the deformation in the gauge theory, is encoded in the pure spinors on the internal six-dimensional manifold. Moreover for the case of the β -deformations, the solution generating T-duality transformation can be seen as a type-changing generalized \hat{B} transform that acts on both the pure spinors and the RR fields. The use of Generalized Complex Geometry is also a necessary ingredient in

studying supersymmetric branes and calibration conditions in flux backgrounds [12]. This should be particularly important in solutions with interesting topologies where BPS states can correspond to branes wrapping non-trivial supersymmetric cycles. For example, using wrapped D3 branes one can determine the exact dimensions of fields in duals of conformal field theories. From this point of view, the simplicity of solutions in section 4, which are characterized by a generalized Kähler potential, suggests that it is perhaps possible to make a general analysis of volumes and R-charges analogous to that in [22]. Moreover, it should be also quite straightforward to verify, using the generalized calibrations provided by the pure spinors, that central and R-charges do not change for marginal deformations.

Finally, this paper is restricted to the study of conformal case, but our equations and formalism have obvious applications to the non conformal case as well.

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A. The general conditions of supersymmetry

In this short appendix we report the condition of supersymmetry corresponding to the most general SU(2) structure ansatz for the spinors (2.4). From equations (2.17) we obtain the following set of differential constraints for the SU(2) invariant forms and the fluxes.

The equation for Φ_- gives

$$d\left(e^{2A-\varphi}(bx - ay)z\right) = 0 \tag{A.1}$$

$$d\left[e^{2A-\varphi}(by\bar{\omega} - ax\omega + (bx + ay)j)z\right] + i(bx - ay)Hz = 0 \tag{A.2}$$

$$d\left[e^{2A-\varphi}(bx - ay)zj^2\right] - 2iHz\left[by\bar{\omega} - ax\omega + (bx + ay)j\right] = 0 \tag{A.3}$$

and that for Φ_+

$$e^{-2A+\varphi}d\left[e^{2A-\varphi}(a\bar{x} + b\bar{y})\right] = dA(x\bar{a} + y\bar{b}) + \frac{e^\varphi}{2}\left[a_-F_1 - ia_+ * F_5\right] \tag{A.4}$$

$$e^{-2A+\varphi}d\left[e^{2A-\varphi}\left((a\bar{x} - b\bar{y})j + \frac{i}{2}(a\bar{x} + b\bar{y})z\bar{z} + a\bar{y}\omega + b\bar{x}\bar{\omega}\right)\right] \tag{A.5}$$

$$-i(a\bar{x} + b\bar{y})H =$$

$$-dA\left[(x\bar{a} - y\bar{b})j + \frac{i}{2}(x\bar{a} + y\bar{b})z\bar{z} + y\bar{a}\bar{\omega} + x\bar{b}\omega\right] + \frac{e^\varphi}{2}\left[ia_-F_3 + a_+ * F_3\right]$$

$$e^{-2A+\varphi}d\left[e^{2A-\varphi}\left((a\bar{x} + b\bar{y})j^2 + i(a\bar{x} - b\bar{y})jz\bar{z} + i(a\bar{y}\omega + b\bar{x}\bar{\omega})z\bar{z}\right)\right] \tag{A.6}$$

$$-2iH\left[a\bar{y}\omega + b\bar{x}\bar{\omega} + (a\bar{x} - b\bar{y})j + \frac{i}{2}(a\bar{x} + b\bar{y})z\bar{z}\right] =$$

$$dA\left[(x\bar{a} + y\bar{b})j^2 + i(x\bar{a} - y\bar{b})jz\bar{z} + i(y\bar{a}\bar{\omega} + x\bar{b}\omega)z\bar{z}\right] - e^\varphi\left[a_-F_5 - ia_+ * F_1\right]$$

with $a_{\pm} = |a|^2 + |b|^2 \pm (|x|^2 + |y|^2)$

Note that the equations of motion for the RR fluxes follow from the pure spinor equations (2.17). However in order to find a complete solution, the NS-flux equation of motion and the Bianchi identities for the fluxes, $dH = 0$ and $(d - H)F = 0$, must still be imposed.

B. Formulae for the T-duality

In this appendix we collect formulae for the explicit action of the T-duality group on various quantities using [20]. The T-duality group of T^2 is $O(2,2) \equiv SL(2,R) \times SL(2,R)$, which can be represented as the set of matrices O such that $O^T J O = J$ where

$$J = \begin{pmatrix} O & I_{2 \times 2} \\ I_{2 \times 2} & O \end{pmatrix}. \tag{B.1}$$

A matrix with two by two blocks

$$O = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{B.2}$$

acts on the two by two matrix $E = g + B$ as [23]

$$E \rightarrow (AE + B)(CE + D)^{-1}. \tag{B.3}$$

The $SL(2,R)$ subgroup that acts on the complexified Kähler modulus of the torus can be parametrized as ¹³

$$O = \begin{pmatrix} aI & b\epsilon \\ -c\epsilon & dI \end{pmatrix}, \tag{B.5}$$

where ϵ is the Pauli matrix $i\sigma_2$ and $ad - bc = 1$.

It is convenient to write

$$O_{LM} = T^{-a} O T^{-a} \tag{B.6}$$

where

$$O = \begin{pmatrix} (S + R)/2 & (S - R)/2 \\ (S - R)/2 & (S + R)/2 \end{pmatrix}, \tag{B.7}$$

with $\gamma = 2a/(1 + a^2)$ and

$$S + R = 2\sqrt{1 - \gamma^2}I, \quad S - R = -2\gamma\epsilon \tag{B.8}$$

¹³It is well known that the second copy of $SL(2,R)$ acts on the complex structure of the two torus

$$M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad O = \begin{pmatrix} M & 0 \\ 0 & M^{T-1} \end{pmatrix}, \quad E \rightarrow MEM^T \tag{B.4}$$

by transforming $g \rightarrow MgM^T$. This is just a change of coordinates and thus less useful for generating new solutions.

and T is the standard shift generator T of $SL(2;\mathbb{R})$

$$T^{-a} = \begin{pmatrix} I_{2 \times 2} & -a\epsilon \\ 0 & I_{2 \times 2} \end{pmatrix}.$$

The advantage of this parametrization is that for matrices of the form (B.7) the T-duality transformations considerably simplify. The part of the transformation corresponding to O is the only one that acts non trivially on the coordinates. T^{-a} just shifts the B field and can be easily superimposed once the action of O is known.

Consider now S, R and O as six-dimensional matrices by trivial extension $\mathcal{R} = (I_{4 \times 4}, R)$ and $\mathcal{S} = (I_{4 \times 4}, S)$. Defining the six dimensional matrix [20]

$$Q = \frac{1}{2}[(\mathcal{S} + \mathcal{R}) + (\mathcal{S} - \mathcal{R})(g_6 + B_6)], \tag{B.9}$$

and the operator

$$\begin{aligned} \Omega_T &= \frac{1}{2} \sqrt{\frac{\det B}{\det Q}} \left(1 - \frac{1}{2} A^{ij} \Gamma_{ij} \right), \\ B^{ij} &= [(\mathcal{R} + \mathcal{S}) + (\mathcal{R} - \mathcal{S})B_6]_{ij}, \\ A^{ij} &= [(\mathcal{S} - \mathcal{R})^{-1}(\mathcal{S} + \mathcal{R}) + B_6]_{ij}^{-1}, \end{aligned} \tag{B.10}$$

the effect of T-duality on the other fields and the supersymmetry parameters can be summarized by [20]

$$g'_6 = Q^{T-1} g_6 Q^{-1}, \tag{B.11}$$

$$e^{\phi'} = \frac{e^\phi}{\det Q}, \tag{B.12}$$

$$F' = \sqrt{\det Q} \Omega_T F$$

$$\eta_+^{1'} = \eta_+^1, \tag{B.13}$$

$$\eta_+^{2'} = \Omega_T \eta_+^2. \tag{B.14}$$

where F , as usual, is the formal sum of RR fields strength, and we also gave an expression for the transformation of the metric alternative to (B.3).

Let us apply these formulae to our case. Starting with flat space and zero B -field, the action of T^{-a} shifts the value of the B -field to $-a$. So we apply the previous formulae for the action of O to a background with $-a$ B -field. By direct computation $\det Q = 1 + \gamma^2 g$ and

$$\Omega_T = \frac{1}{\sqrt{1 + \gamma^2 g}} (1 + \gamma \Gamma_{\varphi_1 \varphi_2}). \tag{B.15}$$

It is also easy to check that the transformation on the metric reproduces equation (5.31).

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